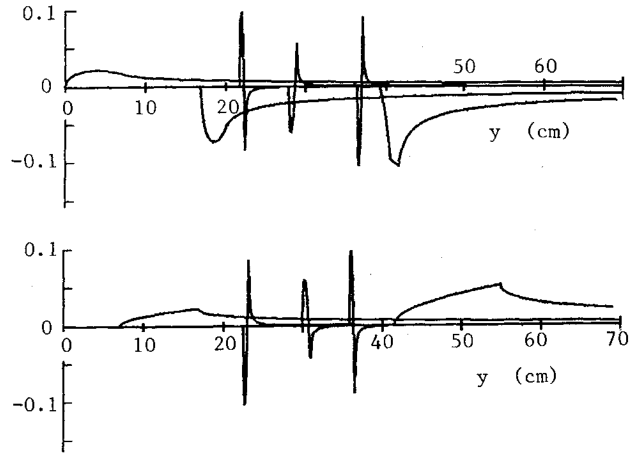

Fig. 6 Individual *F*-functions from unequal length segments for a free-flight projectile ($D = 7.6$ cm, $L = 41$ cm).

at the bumps and groove, $F(y)$ is spiky, characteristic of actual projectiles.

The *F*-functions from individual segments are depicted in Fig. 6. For clarity about half of them are shown at the top and the others appear at the bottom. The contributions from the bumps and groove are interesting. The front of a bump produces a short N-shaped contribution and the rear produces an inverted N-shaped addition. These occur in reverse order for a groove. Such individual contributions are not so apparent in the full $F(y)$ of Fig. 5c.

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Poisson-Kirchhoff Paradox in Flexure of Plates

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Introduction

In a recent article[1] that various plate theories (including his own notable contributions) concerning the Poisson-Kirchhoff boundary conditions paradox, Reissner remarked that the problem of modifying Kirchhoff's approximations of the displacements for an improved sixth-order plate theory has not been completely resolved. He expected that a sixth-order differential equation governing the deflection $w_0(x,y)$ might be of the form $D\nabla^2\nabla^2 w_0 - C\nabla^2\nabla^2\nabla^2 w_0 = q$, with some suitable factor function C. However, no way of finding this equation has been offered until now. In the present work, it is shown that the key to deriving such an equation lies in Reissner's aforementioned remark, with only one slight alteration. Instead of displacements, we direct our attention to the problem of modifying Kirchhoff's assumptions with regard to transverse normal and shear strains.

An iterative scheme is presented here through which an assumed state of transverse normal and shear strains can be transformed into a new state by integration of strain-displacement relations and equilibrium equations. Starting with Kirchhoff's assumptions, the iterative process is carried out to develop expressions for displacements and stresses. In the limit, each of these expressions is in the form of an infinite series in which each term is a product of a known function of thickness coordinate z and a differential expression involving mid-plane deflection $w_0(x,y)$. The expressions thus obtained for displacements and stresses satisfy all field equations for arbitrary function $w_0(x,y)$. By taking the first two terms in the series for normal stress σ_z and satisfying the surface load condition, it is shown that the plate behavior is governed by a sixth-order differential equation involving $w_0(x,y)$. It is indeed, in the form as expected by Reissner.

Analysis

For simplicity in presentation, a rectilinear domain $0 \le x \le a$, $0 \le y \le b$, and $-h \le z \le +h$ with reference to Cartesian coordinate system x,y,z is considered. The thickness $2h$ of the plate is small compared to its lateral dimensions a and b. The material of the plate is homogeneous and isotropic with elastic constants E (Young's modulus), ν (Poisson's ratio), and G (modulus of rigidity) that are related to each other by $E = 2(1+\nu)G$.

The plate is subjected to asymmetric load $q(x,y)$ along top and bottom surfaces ($z = \pm h$). We assume that the boundary conditions along the edges are prescribed so that the in-plane displacements (u,v) and bending stresses $(\sigma_x, \sigma_y, \tau_{xy})$ are antisymmetric in z and the normal deflection w and transverse shear stresses (τ_{xz}, τ_{yz}) are symmetric in z. We treat the problem within the classical small deformation theory of three-dimensional elasticity.

In the iterative scheme envisaged here, Kirchhoff's assumed state of strains is expressed as

$$\epsilon_z^{(0)} = 0, \qquad \gamma_{xz}^{(0)} = 0, \qquad \gamma_{yz}^{(0)} = 0 \tag{1}$$

Let $\epsilon_z^{(i)}$, $\gamma_{xz}^{(i)}$, and $\gamma_{yz}^{(i)}$ be the transverse normal and shear strains at the ith stage of iteration. By substituting in strain-displacement relations,

$$\epsilon_z = \frac{\partial w}{\partial z}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

integrating with respect to z and using the assumed symmetries in the problem, one obtains the displacments in the form

$$w^{(i)} = w_0(x,y) + \int_0^z \epsilon_z^{(i)}\, dz \tag{2}$$

$$u^{(i)} = -\frac{\partial}{\partial x}\left[z w_0 + \int_0^z \int_0^z \epsilon_z^{(i)}\, (dz)_2 \right] + \int_0^z \gamma_{xz}^{(i)}\, dz \tag{3}$$

and a similar expression for $v^{(i)}$.

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In the preceding equations and subsequent equations, the suffix n in $(dz)_n$ denotes the successive integration to be carried out n times with respect to z.

From the displacements $u^{(i)}$, $v^{(i)}$ and the strain-displacement relations

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \qquad \epsilon_x=\frac{\partial u}{\partial x}, \qquad \epsilon_y=\frac{\partial v}{\partial y}$$

the expressions for bending strains are obtained as

$$\gamma_{xy}^{(i)}=-2\frac{\partial^2}{\partial x\partial y}\left[zw_0+\int_0^z\int_0^z\epsilon_z^{(i)}(dz)_2\right]+\int_0^z\left(\frac{\partial}{\partial y}\gamma_{xz}^{(i)}+\frac{\partial}{\partial x}\gamma_{yz}^{(i)}\right)dz \tag{4}$$

$$\epsilon_x^{(i)}=-\frac{\partial^2}{\partial x^2}\left[zw_0+\int_0^z\int_0^z\epsilon_z^{(i)}(dz)_2\right]+\frac{\partial}{\partial z}\int_0^z\gamma_{xz}^{(i)}dz \tag{5}$$

and a similar expression for $\epsilon_y^{(i)}$.

The shear stress-shear strain relations give

$$\tau_{xy}^{(i)}=G\gamma_{xy}^{(i)}, \qquad \tau_{xz}^{(i)}=G\gamma_{xz}^{(i)}, \qquad \tau_{yz}^{(i)}=G\gamma_{yz}^{(i)} \tag{6}$$

If the normal stress components are obtained from the stress-strain relations,

$$\epsilon_x=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)] \tag{7}$$

$$\epsilon_y=\frac{1}{E}[\sigma_y-\nu(\sigma_z+\sigma_x)] \tag{8}$$

$$\epsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)] \tag{9}$$

We notice that in the zeroth approximation in which $\epsilon_z^{(0)}=0$, the normal stresses correspond to a plane strain state whereas the plane stress state is appropriate due to the thinness of the plate. Moreover, the stress components thus obtained have to satisfy the three equilibrium equations

$$\frac{\partial}{\partial x}\sigma_x+\frac{\partial}{\partial y}\tau_{xy}+\frac{\partial}{\partial z}\tau_{xz}=0 \tag{10}$$

$$\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\sigma_y+\frac{\partial}{\partial z}\tau_{yz}=0 \tag{11}$$

$$\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{yz}+\frac{\partial}{\partial z}\sigma_z=0 \tag{12}$$

If the preceding equilibrium equations are to be satisfied by an iterative process with additive corrections to the assumed strains $\epsilon_z^{(i)}$, $\gamma_{xz}^{(i)}$, $\gamma_{yz}^{(i)}$, we find that the determination of these corrective terms would be unwieldy due to the resulting coupled integro-differential equations governing these corrective terms.

Fortunately, the aforementioned difficulties are overcome in a simple manner by adopting the following steps:

1) Obtain σ_z from Eq. (12) by integrating with respect to z.

2) Substitute σ_z thus obtained in Eqs. (7–8) and solve for σ_x and σ_y.

3) Satisfy the constitutive relation (9) and the two in-plane equilibrium equations (10–11) by finding iterative corrections to the assumed transverse strains ϵ_z, γ_{xz}, and γ_{yz}.

From the above procedure, we obtain

$$\sigma_z^{(i)}=-G\int_0^z\left(\frac{\partial}{\partial x}\gamma_{xz}^{(i)}+\frac{\partial}{\partial y}\gamma_{yz}^{(i)}\right)dz \tag{13}$$

$$\sigma_x^{(i)}=-E'\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)\left[zw_0+\int_0^z\int_0^z\epsilon_z^{(i)}(dz)_2\right]+E'\int_0^z\left[\left(1-\frac{\nu}{2}\right)\frac{\partial}{\partial x}\gamma_{xz}^{(i)}+\frac{\nu}{2}\frac{\partial}{\partial y}\gamma_{yz}^{(i)}\right]dz \tag{14}$$

where $E'=E/(1-\nu^2)$ and a similar expression for $\sigma_y^{(i)}$. The new state of strains obtained form step 3 just described is

$$\epsilon_z^{(i+1)}=\frac{\nu}{1-\nu}\nabla^2\left[zw_0+\int_0^z\int_0^z\epsilon_z^{(i)}(dz)_2\right]-\frac{1+2\nu}{2(1+\gamma)}\int_0^z\left(\frac{\partial}{\partial x}\gamma_{xz}^{(i)}+\frac{\partial}{\partial y}\nu_{yz}^{(i)}\right)dz \tag{15}$$

$$\gamma_{xz}^{(i+1)}=-\frac{2}{1-\nu}\frac{\partial}{\partial x}\nabla^2\left[\frac{h^2-z^2}{2}w_0+\int_z^h\int_0^z\int_0^z\epsilon_z^{(i)}(dz)_3\right]+\int_z^h\int_0^z\left[\nabla^2\gamma_{xz}^{(i)}+\frac{1}{1-\gamma}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\gamma_{xz}^{(i)}+\frac{\partial}{\partial y}\nu_{yz}^{(i)}\right)\right](dz)_2 \tag{16}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ and a similar expression for $\gamma^{(i+1)}$. It should be noted that the zero shear + stress conditions at $z=\pm h$ are used in obtaining the expressions for transverse shear strains.

Starting with Kirchhoff's strains Eq. (1) and carrying out the iteration, one obtains in the limit an infinite series expansion for each of the displacements and stresses. A general term in each of these series is of the form $f_n(z)L_n(w_0)$ so that f_{n+1}/f_n is $O(h^2)$ and L_n's are two dimensional differential operators whose orders increase successively by two.

It has also been found that each iterative step gives one additional term in each of the series without affecting the terms in the previous step. Hence, each term in the series can be obtained successively without much difficulty.

At each stage of iteration, errors in satisfying field equations are due to differences in successive expressions for transverse normal and shear strains. In view of the earlier observations about the nature of terms in the series, these errors tend to near zero in the limit. That is, the expressions derived for $u^{(\infty)}$, $v^{(\infty)}$, $w^{(\infty)}$ and stress components $\sigma_x^{(\infty)}$ and so on by the present procedure satisfy all field equations for arbitrary function $w_0(x,y)$.

The middle-plane deflection $w_0(x,y)$ is determined from the surface normal load condition and prescribed boundary conditions along the edges of the plate. The former gives the governing differential equation for $w_0(x,y)$. The latter conditions that vary with z have to be satisfied for all z in the interval $-h\leq z\leq h$. For this purpose, one could use any known procedure for nullification of errors (e.g., least-squares method, Taylor series expansion, method of weighted residuals) and generate necessary boundary conditions for the two-dimensional differential problem governing $w_0(x,y)$.

Here, we restrict ourselves to the derivation of the sixth-order differential equation governing $w_0(x,y)$ that has eluded several investigators over the past eight or nine decades. For this purpose, we take the first two terms in the series previously mentioned for each displacement and stress component. They are given as

$$u=-\left[z+\left(\frac{f_0}{1-\nu}+\frac{\nu}{1-\nu}\frac{z^3}{6}\right)\nabla^2\right]\frac{\partial w_0}{\partial x} \tag{17}$$

$$w=\left(1+\frac{\nu}{1-\nu}\frac{z^2}{2}\nabla^2\right)w_0 \tag{18}$$

$$\sigma_x=-E'\left[z+\left(\frac{f_0}{1-\nu}+\frac{\nu}{1-\nu}\frac{z^3}{6}\right)\nabla^2\right]\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)w_0 \tag{19}$$

$$\tau_{xy}=-E'\left[(1-\nu)z+\left(f_0+\nu\frac{z^3}{6}\right)\nabla^2\right]\frac{\partial^2 w_0}{\partial x\partial y} \tag{20}$$

$$\tau_{xz}=-E'\,[\tfrac{1}{2}f_0'+g_0'\nabla^2]\frac{\partial}{\partial x}\nabla^2 w_0 \tag{21}$$

$$\sigma_z=E'\,[\tfrac{1}{2}f_0+g_0\nabla^2]\,\nabla^4 w_0 \tag{22}$$

where

$$f_0=h^2z-\frac{z^3}{3} \tag{23}$$

$$g_0=\frac{1}{12(1-\nu)}\left[(5-2\nu)h^4z-(2-\nu)h^2z^3+(1-\nu)\frac{z^5}{5}\right] \tag{24}$$

The expressions for v, σ_y, and τ_{yz} can be obtained from u, σ_x, and τ_{xz}, respectively, by interchanging x and y. From Eqs. (22–24) and the surface normal load condition, we obtain

$$\nabla^4 w_0+\frac{8-3\nu}{10(1-\nu)}h^2\nabla^6 w_0-\frac{2q}{D}=0 \tag{25}$$

where

$$D=\frac{2}{3}\frac{Eh^3}{1-\nu^2}$$

is the flexural rigidity of the plate.

With regard to edge conditions, one may use weighted averages and prescribe them, e.g., at $x=$ constant edge

$$1)\ \int_{-h}^{h} uz\,\mathrm{d}z=u^0(y)\quad\text{or}\quad\int_{-h}^{h}\sigma_x z\,\mathrm{d}z=M_x^0(y) \tag{26}$$

$$2)\ \int_{-h}^{h} vz\,\mathrm{d}z=v^0(y)\quad\text{or}\quad\int_{-h}^{h}\tau_{xy}z\mathrm{d}z=M_{xy}^0(y) \tag{27}$$

$$3)\ \int_{-h}^{h} w\,\mathrm{d}z=w^0(y)\quad\text{or}\quad\int_{-h}^{h}\tau_{xz}\,\mathrm{d}z=V_x^0(y) \tag{28}$$

When compared with the classical theory, the second term in Eq. (25) is the correction due to the combined effect of transverse normal and shear strains on bending deformation. The coefficient $k=(8-3\nu)/10(1-\nu)$ is used for comparison with Reissner's theory.

In Reissner's theory, stress resultants along with w_0 are treated as variables and corrective factors due to transverse shear strains[2,3] and, along with normal strain,[4] are associated with stress resultants. For comparison with the present analysis, we consider Reissner's equation[1,2] for w_0 that, in the present notation, is given by

$$\nabla^4 w_0+k_1h^2\nabla^2\left(\frac{2q}{D}\right)-\frac{2q}{D}=0$$

where $k_1=(8-4\nu)/10(1-\nu)$. If we replace $(2q/D)$ by $\nabla^4 w_0$ from Kirchhoff's theory, the equation just given is similar to Eq. (25) except for a small difference $[=\nu/10(1-\nu)]$ in the coefficient of the corrective term. Since Reissner's equation is based on the assumption $\epsilon_z=0$, the difference $(k-k_1)$ in the coefficient is due to the effect of normal strain on bending deformations. Reissner has, however, included the effect of normal strain in the analysis of a later investigation.[4]

Recently, Cheng[5] proposed a refined theory of plates in terms of three fundamental problems. He treated Navier's equations of equlibrium as ordinary differential equations with respect to z and obtained symbolic solutions for the displacements. These symbolic solutions are then used to satisfy free surface conditions for the loading case q equal to zero. By this procedure, a three-dimensional problem is reduced to three two-dimensional problems governed by a biharmonic equation, a shear equation, and a third equation useful for studying edge effects. His expressions for displacements and bending stresses in the first fundamental problem are found to be identical to those in Eqs. (17–20). However, further investigation is required to relate the present theory with the three fundamental problems, particularly the second and third problems, in Cheng's theory.

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